7.0 RATIONAL AND IRRATIONAL NUMBERS.(senior two work)

		CONTENT	COMMENTS
		Objectives: To • Identify rational and irrational numbers.. • Identify surds • Simplifying rational numbers. • Rationalize	
		Key words. Rational numbers, irrational numbers, terminating and recurring decimals, surds, order of surds, conjugate surds, monomial and binomial rationalization,	
	7.1	**Rational numbers.** Are either integers or numbers that can be expressed as a ratio(fraction) of two integers. For example, 3 is a rational number as it can be expressed in the form $3 = \frac{3}{1} = \frac{6}{2}$ $\frac{-5}{7}$ is another rational number. **Terminating decimals** such as 2.5 , 1.2, 0.35 etc. can expressed as a ratio of integers. So, they are ration numbers. **Recurring decimals** can be expressed as ratios of two integers. So, they are rational numbers. **Irrational numbers** cannot be expressed as a ratio of integers. For example, π and roots of certain numbers like $\sqrt{2}$, $\sqrt{3}, \sqrt{5}$ etc.	
	7.2	**Surds**. A **surd** is an irrational root of a rational number. For example, $\sqrt{2}$ is a surd. $\sqrt{9}$ is not a surd whereas $\sqrt[3]{9}$ is a surd. **Order** of a surd is the root of the surd. Square roots are surds of order two. (These are the ones mainly considered in this work). Cube roots are surds of order three. A surd can be **made of one part**. This called a **monomial surd.** For example, $\sqrt{5}$ and $2\sqrt{3}$. While one **with two parts** is called a **binomial surd.** One part can be rational or both parts being irrational. For example, $1 + \sqrt{5}$ and $5 - 2\sqrt{3}$ are binomial surds with the first part being rational and the other, is irrational. $\sqrt{7} + \sqrt{11}$ is also a binomial surd. Notice that both parts are irrational.	
	7.3	**Working with surds.** Surds of the same order can be multiplied.	

$$\sqrt{A} \times \sqrt{B} = \sqrt{AB} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{F 7.1}$$

Thus with the same number, $\sqrt{A} \times \sqrt{A} = \sqrt{A^2} = A$ ……………………F 7.2

Surds of the same order can be divided.

$$\frac{\sqrt{A}}{\sqrt{B}} = \sqrt{\frac{A}{B}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{F 7.3}$$

Surds of the same order and same number under the root sign can be added or subtracted.

$$n\sqrt{A} + m\sqrt{A} = (n+m)\sqrt{A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{F 7.4}$$

And $n\sqrt{A} - m\sqrt{A} = (n-m)\sqrt{A}$ ………………..F 7.5

Example 1.

Express each of the following so that the integer under the root sign is as small as possible.

(a) $\sqrt{18}$

(b) $\sqrt{108}$

(c) $\sqrt{20y^2}$

(a) For $\sqrt{18}$, consider expressing 18 into square number factors.

$$\sqrt{18} = \sqrt{9 \times 2}$$
$$= \sqrt{9} \times \sqrt{2}$$
$$= 3\sqrt{2}$$

(b) Here, 108 = 36×3 (36 being the highest possible square number factor)

$$\sqrt{108} = \sqrt{36 \times 3}$$
$$= \sqrt{36} \times \sqrt{3}$$
$$= 6\sqrt{3}$$

(c) $y^2$is a square factor. Further, 20 = 5 ×4 (4 being a square number)

$$\sqrt{20y^2} = \sqrt{5 \times 4 \times y^2}$$
$$= \sqrt{5} \times \sqrt{4} \times \sqrt{y^2}$$
$$= \sqrt{5} \times 2 \times y = 2y\sqrt{5}$$

Example 2.

Express each of the following as a single number under the root sign.

(a) $2\sqrt{3}$

(b) $\sqrt{2} \times \sqrt{3}$

(c) $2y^2\sqrt{5}$

Where necessary, each of the numbers outside the root sign must be squared to be under the root sign. Otherwise, the product rule(F 7.1) is applied.

(a) $2\sqrt{3} = 2 \times \sqrt{3}$

$= \sqrt{2^2} \times \sqrt{3}$
$= \sqrt{4} \times \sqrt{3}$
$= \sqrt{4 \times 3}$
$= \sqrt{12}$

(b) $\sqrt{2} \times \sqrt{3}$
$= \sqrt{2 \times 3}$
$= \sqrt{6}$

(c) $2y^2\sqrt{5} = 2 \times y^2 \times \sqrt{5}$
$= \sqrt{2^2} \times \sqrt{y^4} \times \sqrt{5}$
$= \sqrt{4} \times \sqrt{y^4} \times \sqrt{5}$
$= \sqrt{4 \times y^4 \times 5}$
$= \sqrt{20y^4}$

Example 3.
Simplify each of the following.

(a) $\frac{\sqrt{72}}{\sqrt{3}}$
(b) $2\sqrt{7} + \sqrt{28}$
(c) $\sqrt{12} - \sqrt{3}$
(d) $2\sqrt{27} + \sqrt{75} - \sqrt{48}$

For each of these questions, it is advisable to express the numbers involved into common surds.

(a) $\frac{\sqrt{72}}{\sqrt{3}} = \frac{\sqrt{24 \times 3}}{\sqrt{3}}$
$= \frac{\sqrt{24} \times \sqrt{3}}{\sqrt{3}}$
$= \sqrt{24}$

(b) $2\sqrt{7} + \sqrt{28}$
$= 2\sqrt{7} + \sqrt{4 \times 7}$
$= 2\sqrt{7} + 2\sqrt{7}$
$= 4\sqrt{7}$

(c) $\sqrt{12} - \sqrt{3}$
$= \sqrt{4 \times 3} - \sqrt{3}$
$= 2\sqrt{3} - \sqrt{3}$
$= \sqrt{3}$

(d) $2\sqrt{27} + \sqrt{75} - \sqrt{48}$
$= 2\sqrt{9 \times 3} + \sqrt{25 \times 3} - \sqrt{16 \times 3}$
$= 2 \times 3\sqrt{3} + 5\sqrt{3} - 4\sqrt{3}$
$= 6\sqrt{3} + 5\sqrt{3} - 4\sqrt{3}$
$= 7\sqrt{3}$

	7.4	**Multiplication of binomial surds.** This can be done by expansion with subsequent collection and simplification of terms.	

Example 4.
Simplify each of the following.

(a) $2(1+\sqrt{3})$
(b) $(2+\sqrt{5})(1+\sqrt{5})$
(c) $(3+\sqrt{2})(2-\sqrt{5})$
(d) $(\sqrt{3}-1)(\sqrt{3}+1)$
(e) $(\sqrt{2}+\sqrt{5})(\sqrt{2}-\sqrt{5})$

(a) $2(1+\sqrt{3})$

$$= 2 + 2\sqrt{3}$$

(b) The second bracket is multiplied by each of the terms in the first bracket.
NOTE: It is advisable to take extra care in opening of the brackets.

Thus $(2+\sqrt{5})(1+\sqrt{5})$

$$= 2(1+\sqrt{5}) + \sqrt{5}(1+\sqrt{5})$$
$$= 2 + 2\sqrt{5} + \sqrt{5} + \sqrt{5}\times\sqrt{5}$$
$$= 2 + 2\sqrt{5} + \sqrt{5} + 5$$
$$= 7 + 3\sqrt{5}$$

(c) $(3+\sqrt{2})(2-\sqrt{5})$

$$= 3(2-\sqrt{5}) + \sqrt{2}(2-\sqrt{5})$$
$$= 6 - 3\sqrt{5} + 2\sqrt{2} - \sqrt{2}\times\sqrt{5}$$
$$= 2 - 3\sqrt{5} + 2\sqrt{2} - \sqrt{2\times 5}$$
$$= 2 - 3\sqrt{5} + 2\sqrt{2} - \sqrt{10}$$

(d) $(\sqrt{3}-1)(\sqrt{3}+1)$

$$= \sqrt{3}(\sqrt{3}+1) - 1(\sqrt{3}+1)$$
$$= \sqrt{3}\times\sqrt{3} + \sqrt{3} - \sqrt{3} - 1$$
$$= 3 - 1$$
$$= 2$$

(e) $(\sqrt{2}+\sqrt{5})(\sqrt{2}-\sqrt{5})$

$$= \sqrt{2}(\sqrt{2}-\sqrt{5}) + \sqrt{5}(\sqrt{2}-\sqrt{5})$$
$$= \sqrt{2}\times\sqrt{2} - \sqrt{2}\sqrt{5} + \sqrt{5}\sqrt{2} - \sqrt{5}\sqrt{5}$$
$$= 2 - 5$$
$$= -3$$

7.5 **Conjugate of binomial surds.**

The conjugate of a binomial surd is one which when multiplied by the surd itself gives a rational number.

Referring to example 4 parts (d) and (e), Notice that the products are rational numbers.

This means that in part

(d), $\sqrt{3}-1$ is the conjugate of $\sqrt{3}+1$ and vice versa.

(e), $\sqrt{2}+\sqrt{5}$ is the conjugate of $\sqrt{2}-\sqrt{5}$ and vice versa.

Example 5.

For each of the following, multiply by its conjugate and simplify your answer.

(a) $2+\sqrt{3}$

(b) $1-3\sqrt{2}$

(c) $\sqrt{2}+3$

(d) $\sqrt{5}-3$

(e) $\sqrt{7}+\sqrt{2}$

(f) $2\sqrt{2}-\sqrt{3}$

(a) $(2+\sqrt{3})(2-\sqrt{3})$

$$= 2(2-\sqrt{3})+\sqrt{3}(2-\sqrt{3})$$
$$= 4-2\sqrt{3}+2\sqrt{3}-\sqrt{3}\times\sqrt{3}$$
$$= 4-2\sqrt{3}+2\sqrt{3}-3$$
$$= 4-3$$
$$= 1$$

(b) $(1-3\sqrt{2})(1+3\sqrt{2})$

$$= 1(1+3\sqrt{2})-3\sqrt{2}(1+3\sqrt{2})$$
$$= 1+3\sqrt{2}-3\sqrt{2}-3\sqrt{2}\times 3\sqrt{2}$$
$$= 1-3\sqrt{2}+3\sqrt{2}-9\times 2$$
$$= 1-18$$
$$= -17$$

(c) $(\sqrt{2}+3)(\sqrt{2}-3)$

$$= \sqrt{2}(\sqrt{2}-3)+3(\sqrt{2}-3)$$
$$= \sqrt{2}\times\sqrt{2}-3\sqrt{2}+3\sqrt{2}-9$$
$$= 2-3\sqrt{2}+3\sqrt{2}-9$$
$$= 2-9$$
$$= -7$$

(d) $(\sqrt{5}-3)(\sqrt{5}+3)$

$$= \sqrt{5}(\sqrt{5}+3)-3(\sqrt{5}+3)$$
$$= \sqrt{5}\times\sqrt{5}+3\sqrt{5}-3\sqrt{5}-9$$

Students may discover a shorter way of working out.

		$= 5 + 3\sqrt{5} - 3\sqrt{5} - 9$ $= 5 - 9$ $= -4$ (e) $(\sqrt{7} + \sqrt{2})(\sqrt{7} - \sqrt{2})$ $= \sqrt{7}(\sqrt{7} - \sqrt{2}) + \sqrt{2}(\sqrt{7} - \sqrt{2})$ $= \sqrt{7} \times \sqrt{7} - \sqrt{7}\sqrt{2} + \sqrt{7}\sqrt{2} - \sqrt{2}\sqrt{2}$ $= 7 - 2$ $= 5$ (f) $(2\sqrt{2} - \sqrt{3})(2\sqrt{2} + \sqrt{3})$ $= 2\sqrt{2}(2\sqrt{2} + \sqrt{3}) - \sqrt{3}(2\sqrt{2} + \sqrt{3})$ $= 2\sqrt{2} \times 2\sqrt{2} + 2\sqrt{2}\sqrt{3} - 2\sqrt{3}\sqrt{2} + \sqrt{3}\sqrt{3}$ $= 4 \times 2 - 3$ $= 8 - 3$ $= 5$	
	7.6	**Rationalizing**. Rationalizing is changing an irrational denominator of a fraction into a rational one. This can be done by multiplying through by the: - surd in the denominator where there is a monomial denominator. - conjugate of the denominator where there is a binomial denominator. ***Example 6.*** *Rationalize:* $(a) \frac{1}{\sqrt{2}}$ $(b) \frac{\sqrt{3}}{\sqrt{2}}$ $(c) \frac{\sqrt{5}}{7\sqrt{3}}$ (a) $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ $= \frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ $= \frac{\sqrt{3} \times \sqrt{2}}{2}$ $= \frac{\sqrt{6}}{2}$	

(c) $\frac{\sqrt{5}}{7\sqrt{3}}$

$$= \frac{\sqrt{5}}{7\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{\sqrt{5\times3}}{7\times3}$$
$$= \frac{\sqrt{15}}{21}$$

Example 7.
Rationalize.

(a) $\frac{1}{\sqrt{3}+1}$
(b) $\frac{2}{\sqrt{2}-1}$
(c) $\frac{1}{2-\sqrt{2}}$
(d) $\frac{\sqrt{3}}{2-\sqrt{5}}$
(e) $\frac{4}{\sqrt{2}+\sqrt{5}}$

(a) $\frac{1}{\sqrt{3}+1}$

$$= \frac{1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$
$$= \frac{\sqrt{3}-1}{3-1}$$
$$= \frac{\sqrt{3}-1}{2} \text{ or } \frac{\sqrt{3}}{2} - \frac{1}{2}$$

(b) $\frac{2}{\sqrt{2}-1}$

$$= \frac{2}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$$
$$= \frac{2(\sqrt{2}+1)}{2-1}$$
$$= \frac{2\sqrt{2}+2}{1}$$
$$= 2\sqrt{2} + 2$$

(c) $\frac{1}{2-\sqrt{2}}$

$$= \frac{1}{2-\sqrt{2}} \times \frac{2+\sqrt{2}}{2+\sqrt{2}}$$
$$= \frac{2+\sqrt{2}}{4-2}$$
$$= \frac{2+\sqrt{2}}{2} \text{ or } 1+ \frac{\sqrt{2}}{2}$$

(d) $\frac{\sqrt{3}}{2-\sqrt{5}}$

$$= \frac{\sqrt{3}}{2-\sqrt{5}} \times \frac{2+\sqrt{5}}{2+\sqrt{5}}$$
$$= \frac{\sqrt{3}(2+\sqrt{5})}{4-5}$$

$$= \frac{2\sqrt{3}+\sqrt{3}\times\sqrt{5}}{1}$$
$$= 2\sqrt{3} + \sqrt{15}$$

(e) $\frac{4}{\sqrt{2}+\sqrt{5}}$

$$= \frac{4}{\sqrt{2}+\sqrt{5}} \times \frac{\sqrt{2}-\sqrt{5}}{\sqrt{2}-\sqrt{5}}$$
$$= \frac{4(\sqrt{2}-\sqrt{5})}{2-5}$$
$$= \frac{4\sqrt{2}-4\sqrt{5}}{-3}$$

Example 8

Rationalize:

(a) $\frac{2}{3\sqrt{2}-\sqrt{5}}$

(b) $\frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}-\sqrt{5}}$

(a) $\frac{2}{3\sqrt{2}-\sqrt{5}}$

$$= \frac{2}{3\sqrt{2}-\sqrt{5}} \times \frac{3\sqrt{2}+\sqrt{5}}{3\sqrt{2}+\sqrt{5}}$$
$$= \frac{2(3\sqrt{2}+\sqrt{5})}{18-5}$$
$$= \frac{6\sqrt{2}+2\sqrt{5}}{13}$$

(b) $\frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}-\sqrt{5}}$

$$= \frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}-\sqrt{5}} \times \frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}+\sqrt{5}}$$
$$= \frac{\sqrt{2}(\sqrt{2}+\sqrt{5})+\sqrt{5}(\sqrt{2}+\sqrt{5})}{2-5}$$
$$= \frac{\sqrt{2}\times\sqrt{2}+\sqrt{2}\times\sqrt{5}+\sqrt{5}\times\sqrt{2}+\sqrt{5}\times\sqrt{5}}{-3}$$
$$= \frac{2+\sqrt{10}+\sqrt{10}+5}{-3}$$
$$= \frac{7+2\sqrt{10}}{-3}$$

Example 9.

Simplify:

(a) $\frac{1}{\sqrt{2}+\sqrt{5}}+\frac{2}{\sqrt{2}-\sqrt{5}}$

(b) $\frac{\sqrt{63}+\sqrt{28}}{\sqrt{175}-\sqrt{63}}$

(c) $\frac{\sqrt{30}+\sqrt{35}}{\sqrt{6}+\sqrt{7}}$

(a) $\frac{1}{\sqrt{2}+\sqrt{5}}+\frac{2}{\sqrt{2}-\sqrt{5}}$

Rationalizing each term,

$$=\frac{1\times(\sqrt{2}-\sqrt{5})}{(\sqrt{2}+\sqrt{5})(\sqrt{2}-\sqrt{5})}+\frac{2\times(\sqrt{2}+\sqrt{5})}{(\sqrt{2}-\sqrt{5})(\sqrt{2}+\sqrt{5})}$$
$$=\frac{\sqrt{2}-\sqrt{5}}{2-5}+\frac{2\sqrt{2}+2\sqrt{5}}{2-5}$$
$$=\frac{\sqrt{2}-\sqrt{5}+2\sqrt{2}+2\sqrt{5}}{-3}$$
$$=\frac{3\sqrt{2}+\sqrt{5}}{-3}$$

(b) $\frac{\sqrt{63}+\sqrt{28}}{\sqrt{175}-\sqrt{63}}$

Notice that the large figures are undesirable. Some simplification may be useful.

Thus, $\frac{\sqrt{63}+\sqrt{28}}{\sqrt{175}-\sqrt{63}}$

$$=\frac{\sqrt{9\times7}+\sqrt{4\times7}}{\sqrt{25\times7}-\sqrt{9\times7}}$$
$$=\frac{3\sqrt{7}+2\sqrt{7}}{5\sqrt{7}-3\sqrt{7}}$$
$$=\frac{5\sqrt{7}}{2\sqrt{7}}$$
$$=\frac{5}{2}$$

(c) $\frac{\sqrt{30}+\sqrt{35}}{\sqrt{6}+\sqrt{7}}$

$$=\frac{\sqrt{5\times6}+\sqrt{5\times7}}{\sqrt{6}+\sqrt{7}}$$
$$=\frac{\sqrt{5}\times\sqrt{6}+\sqrt{5}\times\sqrt{7}}{\sqrt{6}+\sqrt{7}}$$
$$=\frac{\sqrt{5}(\sqrt{6}+\sqrt{7})}{\sqrt{6}+\sqrt{7}}$$
$$=\sqrt{5}$$

Exercise

1. *Simplify and give your answer in the form* $a\sqrt{b}$.

   (a) $\sqrt{12} - \sqrt{3}$

   (b) $2\sqrt{45} - \sqrt{20} + \sqrt{80}$

   (c) $(3 + \sqrt{2})^2$

2. *If* $1 + \sqrt{3} = 2.7321$, *find the value of:*

   (a) $2 + \sqrt{12}$ *to four decimal places.*

   (b) $\sqrt{16} + \sqrt{48}$ *to four decimal places.*

3. *Express each of the following with a rational denominator.*

   (a) $\frac{2}{1+\sqrt{3}}$

   (b) $\frac{\sqrt{7}}{2-3\sqrt{7}}$

   (c) $\frac{2-\sqrt{3}}{2+\sqrt{3}}$

   (d) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{8}}$

4. *If* $\frac{2}{4+\sqrt{5}} = a + b\sqrt{c}$, *find the values of a, b and c.*

5. *Without using a calculator of mathematical tables, find the value of* $\frac{\sqrt{18}+\sqrt{8}}{\sqrt{50}-\sqrt{18}}$ *to one decimal place.*

6. *(a) Express* $\frac{1+\sqrt{5}}{\sqrt{5}-2}$ *in the form* $a + b\sqrt{c}$.

   (b) Hence, if $\sqrt{5} = 2.2361$, *evaluate* $\frac{1+\sqrt{5}}{\sqrt{5}-2}$*correct to four significant figures.*